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Generation of exact analytic bound state solutions from solvable non-powerlaw potentials by a transformation method

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Abstract. A transformation method has been applied to the exactly solvable Hulthen problem to generate a hierarchy of exactly solved quantum systems in any chosen dimension. The generated quantum systems are, in general, energy-dependent with a single normalized eigenfunction, as the Hulthen potential is a non-powerlaw potential. A method has been devised to convert a subset of the generated quantum systems with energy-dependent potentials to a single normal system with an energy-independent potential that behaves like a potential qualitatively similar to the Poschl-Teller potential. A second-order application of the transformation method on the Hulthen system produces another Sturmian quantum system and a different method is given to regroup them into a normal quantum system which resembles the Morse potential. Existence of normalizable eigenfunctions for these systems are found to be dependent on the local and asymptotic behaviour of the transformation function.

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1 Introduction

Quantum mechanics owes its enormous success to its widespread applications to different physical microsystems. The success in obtaining information about the microsystems largely relied on the various approximation schemes such as perturbation theory, variational technique, WKB method etc. This is necessitated as the potential that governs a given quantum system (QS) more often than not, does not facilitate exact solution of the Schrödinger equation. However, for accuracy and ease in analysis within the framework of an approximation schemes it is necessary that an exactly solvable potential may be had which differs from the given potential by as little as possible. This warrants acquisition of as many exactly solvable potentials as possible. Till a decade or two ago the list of exactly solvable potentials was quite meagre. Different authors have reported various solvable potentials some of which with appended conditions. These are quasi-exactly solvable QS [1–5], conditionally exactly solvable QS [6,7] and conditionally quasi-exactly solvable QS [8]. Super symmetric (SUSY) quantum mechanics can also be utilized to generate isospectral solvable potentials [9]. In order to increase this set of exactly solvable potentials, one of the present authors [10] has devised a transformation scheme to generate new exactly solvable potentials from already known non-relativistic exact solutions of QS. The method is based on a transformation called the extended transformation (ET) that includes a co-ordinate transformation (CT) followed by a functional transformation (FT) and a set of plausible ansatz. As shown in the paper [10], the dimension of the (Euclidean) space into which the transformed system gets transported to, in the case of power-law potentials can be arbitrarily preassigned only when ET is performed, although CT is the basic transformation. On the other hand application of CT alone on the non-powerlaw potentials make the situation more complicated as the transformed equation cannot be put in the standard Schrödinger equation form. It is therefore imperative to apply ET in the case of non-powerlaw potential. In this paper, we apply this transformation method to the exactly solvable Hulthen problem, taking it as a typical representative of non-powerlaw potential, which generates a spectrum of different solved QSs, each equipped with normalized exact analytic solutions and associated energy eigenvalues. We discuss a procedure to regroup this set of energy-dependent Sturmian QS to a normal QS. This normal QS, called B-Quantum system (B-QS) is found to be qualitatively similar but analytically different from the Poschl-Teller potential.

Application of ET on a QS with powerlaw potential generates a QS with a new powerlaw potential. Successive application of ET on the generated QS will revert it back to the parent QS. As an example, application of ET on

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the coulomb system will generate the harmonic oscillator (HO) system and application of ET on the HO system will give us back the coulomb system. But in the case of non-powerlaw potentials, ET may be applied repeatedly by selecting the "working potential" differently from the multi-term potential to generate a variety of new QSs except for one which reverts back to the (immediate) parent QS. To demonstrate this, we select one of the terms of the newly generated exactly solved potentials of B-QS as the working potential which specifies the form of transformation and then performing a subsequent transformation we can generate another new set of Sturmian QSs. As there is no standard procedure to regroup them to get a normal QS, we have to use different QS-specific regrouping techniques to produce a normal QS, which we call C-quantum system (C-QS), having exact normalized eigenfunctions and energy independent potential. Here we find the new potential so obtained to be similar to the Morse potential. The above procedure can be repeated to generate more QSs.

In the present paper, our main objective is to generate solvable potentials and to show their hierarchal connections, since exactly solvable potentials facilitate physical applicabilities. Our intention of mentioning, besides other potentials, the Poschl-Teller and the Morse-like potentials as the potentials that can be obtained by ET starting from exactly solvable non-powerlaw Hulthen potential is to show that within the context of ET method they belong to the same family.

The organization of the paper is as follows: in Section 2 we give the formalism of the transformation method modified to tackle multiterm potentials, and the normalizability of the generated eigenfunctions is discussed in Section 3. In Section 4, application of the transformation method to the Hulthen system along with generation of a Sturmian type of QS is given. The method of regrouping the set of energy-dependent QS's to a normal QS is also discussed in the same section. Second-order application of the transformation method on the Hulthen QS is given in Section 5 along with another regrouping method to make normal QS from Sturmian QS's.

2 Formalism

Unlike the powerlaw potentials, for non-powerlaw potentials exact solutions are available only for s-waves. The radial part of the Schrödinger equation for an exactly solved quantum system, with central non-powerlaw potential $V(r)$, henceforth called the A-system in D_A-dimensional Euclidean space [11] is ($\hbar = 1 = 2m$):

$$\psi''_A(r) + \frac{D_A-1}{r}\psi'_A(r) + \left[E_n^A - V_A(r)\right]\psi_A(r) = 0 \quad (1)$$

where the normalized eigenfunctions $\psi_A(r)$ and eigenenergies E_n^A are known for the given $V_A(r)$. Prime denotes the differentiation of the function with respect to its argument.

Applying the extended transformation to equation (1), which comprises of

$$r \rightarrow g_B(r) \quad (2)$$

and

$$\psi_B(r) = f_B^{-1}(r)\psi_A(g_B(r)) \quad (3)$$

leads to the following equation:

$$\psi''_B(r) + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f_B^2 g^{D_A-1}}{g'_B}\right)\psi'_B(r) + \Bigg[\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln f_B\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f'_B g_B^{D_A-1}}{g'_B}\right) + g'^2_B\left(E_n^A - V_A(g_B)\right)\Bigg]\psi_B(r) = 0 \quad (4)$$

where $g_B(r)$ is the transformation function which is at least three times differentiable and f_B^{-1} is a modulated amplitude. The dimension of the Euclidean space of the transformed quantum system (henceforth called the B-Sturmian Quantum systems (B-SQS)) can now be chosen arbitrarily. Let it be denoted by D_B. Then

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{f_B^2 g_B^{D_A-1}}{g'_B} = \frac{D_B-1}{r},$$

which fixes $f_B(r)$ as a function of $g_B(r)$ and its derivative:

$$f_B(r) = g'^{\frac{1}{2}}_B g_B^{-\frac{D_A-1}{2}} r^{\frac{D_B-1}{2}}. \quad (5)$$

In order to reduce equation (4) to the standard Schrödinger equation form, the following plausible ansatz have to be made, which are an integral part of the transformation method:

$$g'^2_B(r)V_A(g_B(r)) = -K_B^2 \quad (6)$$

$$g'^2_B(r)E_n^A = -V_B(r) + E_N^B - K_B^2 \quad (7)$$

where K_B^2 is a constant independent of r. Now $V_A(g_B(r))$, known in terms of $g_B(r)$, specifies the form of the transformation function $g_B(r)$ through equation (6). Equation (7) specifies the potential of the B-SQS once $g_B(r)$ becomes known from relation (6). Furthermore equation (7) yields the energy eigenvalue E_N^B.

The $V_A(g_B(r))$ in equation (6) is termed as the working potential which specifies $g_B(r)$ and is assumed to be monoterm. In the case where the system A has a multiterm potential,

$$V_A(g_B(r)) = V_A^{(1)}(g_B) + V_A^{(2)}(g_B) + \cdots + V_A^{(n)}(g_B) \quad (8)$$

the working potential can be chosen in principle in (2^n-1) different ways. In fact, we can pick any number of terms of the multiterm potential, the least being a single-term and designate it as the working potential $V^{(w)}(g_B(r))$. This necessitates modifications of the ansatz (6) and (7) as:

$$g_B'^2(r)V_A^{(w)}(g_B(r)) = -K_B^2 \tag{9}$$

$$g_B'^2 E_n^A = -V_B^{(1)}(r) + E_N^B - K_B^2 \tag{10}$$

$$g_B'^2[V_A(g_B) - V_A^{(w)}(g_B)] = -V_B^{(2)}(r). \tag{11}$$

Invoking the ansatz (6) and (7) on equation (4), we get the standard Schrödinger equation of the B-SQS:

$$\begin{aligned}\psi_B''(r) + \frac{D_B-1}{r}\psi_B'(r) + \Bigg\{E_N^B - \Big(V_B^{(1)}(r) + V_B^{(2)}(r)\Big) \\ - \Bigg[-\frac{1}{2}\{g_B, r\} + \frac{(D_A-1)}{2}\frac{(D_A-3)}{2}\frac{g_B'^2}{g_B^2} \\ - \frac{(D_B-1)}{2}\frac{(D_B-3)}{2}\frac{1}{r^2}\Bigg]\Bigg\}\psi_B(r) = 0\end{aligned} \tag{12}$$

where

$$\{g_B, r\} = \frac{g_B'''(r)}{g_B'(r)} - \frac{3}{2}\frac{g_B''^2(r)}{g_B'^2(r)}$$

is the Schwartzian derivative symbol. The final form of the Schrödinger equation of the B-system established in an Euclidean space of the chosen dimension D_B is:

$$\psi_B''(r) + \frac{D_B-1}{r}\psi_B'(r) + \left[E_N^B - V_B(r)\right]\psi_B(r) = 0 \tag{13}$$

where

$$V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r) \tag{14}$$

and

$$\begin{aligned}V_B^{(3)}(r) = \frac{(D_A-1)(D_A-3)}{4}\frac{g_B'^2}{g_B^2} \\ - \frac{(D_B-1)(D_B-3)}{4}\frac{1}{r^2} - \frac{1}{2}\{g_B, r\}\end{aligned} \tag{15}$$

whose eigenfunction is given by equations (3, 5):

$$\psi_B(r) = g_B'^{-\frac{1}{2}}(r)\, g_B^{\frac{D_A-1}{2}}\, r^{-\frac{D_B-1}{2}}\,\psi_A(g_B(r)) \tag{16}$$

and is known, since $\psi_A(r)$ and $g_B(r)$ are known.

The set of energy eigenvalues E_N^B of the B-SQS is simply obtained by putting the r-independent overall factor of $V_B^{(1)}(r)$ which would be a product of a function of the form $F(E_N^B)$ and E_n^A, equal to constant c_B^2 which is in fact a characteristic constant of the B-SQS. This characteristic constant c_B^2 plays the same role as $-Ze^2$ in case of coulomb and $mw^2/2$ in the case of the HO system. We write

$$V_B^{(1)}(r) = F(E_N^B)E_n^A v(r) = c_B^2 v(r) \tag{17}$$

where $v(r)$ is the residual r-dependent factor of $V_B^{(1)}(r)$. This yields the eigenvalues

$$E_N^B = F^{-1}\left(\frac{c_B^2}{E_n^A}\right) \tag{18}$$

and is known, as the eigenenergies E_n^A are known. In non-powerlaw cases, in general, $F(E_N^B)$ cannot be factored out from $V_B^{(1)}(r)$. The generated QS would be a Sturmian QS. In order to make a normal QS, we require a QS-specific regrouping technique as mentioned below.

Here we may mention that the transformation procedure based on a co-ordinate transformation is found to be inadequate as it leads to problems regarding dimensionality of the Euclidean space into which the transformed system gets transported to. For example, when we start from a 3-dimensional powerlaw system and apply co-ordinate transformation, the dimension of the transformed quantum system may be a fractional one leading to an (as yet) unphysical quantum system. Moreover, the transformed systems cannot be cast in the standard Schrödinger equation form when the potential is non-powerlaw. The difficulty arising due to the dimensional mismatch can be overcome by performing an extended transformation instead of the simple co-ordinate transformation on the A-system. This extended transformation also solves the problem encountered in applying the co-ordinate transformation to the Schrödinger equation with a non-powerlaw potential. The functional transformation component of the extended transformation may be considered as a device for dimensional reduction or dimensional extension of the generated QSs.

3 Normalizability of generated eigenfunctions

A very useful property of the transformation method that we would like to note is that the wavefunctions of the generated quantum systems are almost always normalizable. Normalizability condition for D_B-dimensional B-QS eigenfunction is:

$$\int_0^\infty \psi_B^2(r) r^{D_B-1}\mathrm{d}r = \frac{1}{|N_B|^2} = \text{finite}$$

using equations (2, 3, 5) we have

$$\psi_B(r) = g_B'^{-\frac{1}{2}}\, g_B^{\frac{D_A-1}{2}}\, r^{-\frac{D_B-1}{2}}\,\psi_A(g_B(r)).$$

From equation (9) it reduces to

$$|N_B|^2\frac{\langle V_A(r)\rangle}{-E_N^B} = 1, \tag{19}$$

since

$$g_B'^2 = \frac{-E_N^B}{V_A(g_B(r))}.$$

Hence all the $\psi_B(r)$ are normalizable for which $E_N^B \neq 0$. For any real QS, $\langle V_A(r)\rangle$ exists. As such, the wavefunctions of the generated QS are always normalizable corresponding to non-null eigenenergies, when the wavefunctions of the parent QS are normalizable. Hence the wavefunctions of the C-QS are also normalizable. The above expressions show that it is a positive feature of the transformation procedure, that more often than not the wavefunction of the newly generated QS is normalizable. This implies that, unless $g(r)$ is not badly behaved, so far as its local and asymptotic properties are concerned, the transformation method carries over the normalizability property of the parent QS to the daughter QSs.

4 First-order transformation

We consider the Hulthen QS [13] as a typical representative of a QS with non-powerlaw potential which is exactly solvable for $l = 0$ case only. Let it be denoted as our A-system. The radial Schrödinger equation in D_A-dimension ($D_A = 3$) is:

$$\psi_A''(r) + \frac{2}{r}\psi_A'(r) + \left(-\alpha^2 - V_A(r)\right)\psi_A(r) = 0 \qquad (20)$$

where

$$V_A(r) = -\beta^2 \frac{e^{-\frac{r}{a}}}{1 - e^{-\frac{r}{a}}} \qquad (21)$$

is the Hulthen potential. The s-state energy eigenfunctions are:

$$\psi_A(r) = \frac{N_A}{r}\exp\left(-\frac{\alpha r}{a}\right) \times {}_2F_1\left(-n, 2\alpha + n, 2\alpha + 1; \exp\left(-\frac{r}{a}\right)\right) \qquad (22)$$

where $\alpha^2 = -E_n^A > 0$; $\beta^2 = V_0 > 0$, the normalization constant is:

$$N_A = \frac{[2\alpha(\alpha + n)\Gamma(2\alpha + n)\Gamma(2\alpha + n + 1)]^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}[\Gamma(n + 1)\Gamma(2\alpha + 1)]} \qquad (23)$$

and energy eigenvalues are,

$$E_n^A = -V_0\left(\frac{\beta^2 - n^2}{2n\beta}\right)^2 \qquad (24)$$

with $\beta^2 > n^2, (n = 1, 2, 3, ...)$.

Now applying extended transformation on equation (20), we obtain the Schrödinger equation for the new quantum system for $D_B = 3$ as,

$$\psi_B''(r) + \frac{2}{r}\psi_B'(r) + \Big[\frac{1}{2}\{g_B, r\} + g_B'^2\left(-\alpha^2 - V_A(g_B)\right)\Big]\psi_B(r) = 0. \qquad (25)$$

Ansatz (6) and (7) give

$$g_B'^2 \frac{\mathrm{e}^{-\frac{g_B}{a}}}{\left(1 - \mathrm{e}^{-\frac{g_B}{a}}\right)} = \frac{K_B^2}{\beta^2} \qquad (26)$$

$$-g_B'^2\alpha^2 = -V_B^{(1)}(r) + E_N^B - K_B^2 \qquad (27)$$

$$V_B^{(3)}(r) = -\frac{1}{2}\{g_B, r\} \qquad (28)$$

where K_B^2 is a constant independent of r. The functional form of $g_B(r)$ obtained from equation (26) by integration is:

$$g_B(r) = 2a \ln \sec \eta r \qquad (29)$$

where

$$\eta = \frac{1}{2a}\sqrt{\frac{K_B^2}{\beta^2}}\cdot \qquad (30)$$

The integration constant is put equal to zero which attributes the local property $g_B(0) = 0$. This local property is desirable from the point of view of normalizability of the wave function of the generated QS.

Now equations (27, 29) lead to

$$V_B^{(1)}(r) = 4a^2\eta^2\alpha^2 \tan^2 \eta r. \qquad (31)$$

Using equations (30, 31) can be written as

$$V_B^{(1)}(r) = K_B^2\left(\frac{\beta^2 - n^2}{2n\beta}\right)^2 \tan^2 \eta r = c_B^2 \tan^2 \eta r, \qquad (32)$$

where c_B is the characteristic constant for the B-SQS [5] and is

$$c_B^2 = K_B^2\left(\frac{\beta^2 - n^2}{2n\beta}\right)^2\cdot \qquad (33)$$

The multiterm potential of the B-SQS, given by equation (14) becomes

$$V_B(r) = 3\eta^2 \csc^2 2\eta r - \eta^2 \sec^2 \eta r + c_B^2 \tan^2 \eta r \qquad (34)$$

which specifies a Sturmian QS. For our purpose, without loss of generality we take for convenience a scale factor $a = 1$. Since β was originally a n-independent parameter, equation (30) shows that K_B or equivalently η has a dependence on the positive integer n. Furthermore by equation (27) $E_N^B = K_B^2$ as $g_B(r)$ is r-dependent and α is a constant. The expression (33) is utilized to find the energy eigenvalues of the Sturmian B-SQS, and is,

$$E_N^B = c_B^2\left(\frac{2n\beta}{\beta^2 - n^2}\right)^2\cdot \qquad (35)$$

Invoking the ansatz (26–28) in equation (20), the radial Schrödinger equation for the B-SQS takes the form:

$$\psi_B''(r) + \frac{2}{r}\psi_B'(r) + (E_N^B - V_B(r))\psi_B(r) = 0 \qquad (36)$$

where the exact eigenfunction $\psi_B(r)$ is given by equation (16) and is,

$$\psi_B(r) = \frac{N_B}{r}\cos^{2\alpha} \eta r(\cot \eta r)^{\frac{1}{2}} \times {}_2F_1(-n, 2\alpha + n, 2\alpha + 1; \cos^2 \eta r)$$

where

$$N_B = \frac{\Gamma(2\alpha+n)}{\Gamma(2\alpha+1)}\left(\frac{2\eta(\alpha+n)(2\alpha+n)}{\Gamma(n)\Gamma(n+1)}\right)^{\frac{1}{2}} \tag{37}$$

$V_B(r)$ is n-dependent, *i.e.*, Sturmian type of potential as α is the energy eigenvalue of the A-system. This special type of energy-dependent potential is equipped with only a single normalized eigenstate *i.e.*, n ceases to be a quantum number. It rather behaves like a system index enumerating different QSs. The B-SQS actually comprises of a finite set of quantum systems. The number of such systems depends on the magnitude of V_0 as we require that $0 < n^2 < V_0$.

The B-SQS can be converted to a normal quantum system by reversing the roles of α and β. We have the A-system relation $\beta^2 = n(2\alpha+n)$ which prohibits α and β both to be n-independent parameters simultaneously. As such, in the B-SQS we take α to be n-independent which makes β n-dependent automatically. To make η n-independent we further take $K_B = 2\beta s$ where a scale factor s is introduced. This makes the potential $V_B(r)$ n-independent and is:

$$V_B(r) = s^2[3\csc^2 2sr - \sec^2 sr + 4\alpha^2\tan^2 sr] \tag{38}$$

and the quantized energy eigenvalues of this normal quantum system (B-QS) becomes,

$$E_n^B = 4s^2 n(2\alpha+n). \tag{39}$$

It may be mentioned that $E - V(r)$ for the Poschl-Teller [14] potential is,

$$\left[2V_0 n(n+\chi+\lambda) + \frac{1}{2}V_0(\chi+\lambda)^2\right] - \frac{1}{2}V_0\left[\frac{\chi(\chi-1)}{\sin^2\alpha r} + \frac{\lambda'(\lambda'-1)}{\cos^2\alpha r}\right]$$

which is exactly same as $E_n^B - V_B(r)$ except that for the B-QS $\chi = 3/2, \lambda \to \lambda' = \lambda + 2$.

So the generated B-QS belongs to the family of the Poschl-Teller potential.

5 Second-order transformation

Application of the extended transformation on the B-SQS comprising of equations (34, 36) we generate another new Sturmian quantum system, say, the C-SQS given by:

$$\psi_c''(r) + \frac{2}{r}\psi_c'(r) + \Big(\frac{1}{2}\{g_c, r\} + g_c'^2\left(E_n^B - V_B(g_c)\right)\Big)\psi_c(r) = 0. \tag{40}$$

The exact energy eigenfunction is

$$\psi_c(r) = \frac{1}{r}g_c(r)g_c'^{-1/2}\psi_B(g_c(r)) \tag{41}$$

and is completely specified once $g_c(r)$ is known. The B-SQS potential is a multiterm potential and we have a number of choices to select the working potential. As a specific choice we consider $-\eta^2\sec^2\eta g_c(r)$ and the set of ansatz required to bring equation (40) into the standard Schrödinger equation form are now,

$$g_c'^2 V_B^{(w)}(g_c) = -K_c^2 \tag{42}$$

$$V_B^{(w)}(g_c) = -\eta^2\sec^2 g_c(r) \tag{43}$$

$$\frac{1}{2}\{g_c, r\} + g_c'^2(E_n^B - 3\eta^2\csc^2 2\eta g_c - 4\eta^2\alpha^2\tan^2\eta g_c) = E_n^c - K_c^2 - V_c(r). \tag{44}$$

From equations (42–44) the new Schrödinger equation for the C-SQS can be written as,

$$\psi_c''(r) + \frac{2}{r}\psi_c'(r) + (E_n^c - V_c(r))\psi_c(r) = 0 \tag{45}$$

where E_n^c is the energy eigenvalue and $V_c(r)$ is the C-SQS potential.

From equation (42) we calculate $g_c(r)$ by integration which is

$$g_c(r) = \frac{1}{\eta}\arccos(\mathrm{sech}K_c r) = \frac{1}{\eta}\arcsin(\tanh K_c r) \tag{46}$$

with the local property, $g_c(0) = 0$.

This yields the energy eigenvalues, the n-dependent potential and the normalized energy eigenfunctions of the C-SQS as:

$$E_n^c = 4K_c^2 n(2\alpha+n) - \frac{K_c^2}{4} \tag{47}$$

$$V_c(r) = K_c^2\left(\left\{4(\alpha+n)^2 - \frac{1}{4}\right\}\tanh^2 K_c r + \frac{3}{4}(\sinh K_c r)^{-2}\right) \tag{48}$$

and

$$\psi_c(r) = \frac{N_c}{r}\frac{1}{(\tanh K_c r)^{\frac{1}{2}}}\mathrm{sech}^{2\alpha}K_c r \times {}_2F_1(-n, 2\alpha+n, 2\alpha+1; \mathrm{sech}^2 K_c r). \tag{49}$$

respectively and the normalization constant is

$$N_c = \frac{2\Gamma(2\alpha+n)}{\Gamma(2\alpha+1)\Gamma(n+1)}\{\alpha n(2\alpha+n)\}^{\frac{1}{2}}. \tag{50}$$

To convert the C-SQS into a normal quantum system we use the following ansatz:

$$(\alpha+n)^2 = c_c^2 \tag{51}$$

c_c^2 being the characteristic constant of the C-QS, which may take both positive and negative values.

The equation (51) gives

$$\begin{aligned} \alpha &= \pm c_c - n, \;\; c_c^2 > 0 \\ &= \pm \mathrm{i} c_c - n, \;\; c_c^2 < 0. \end{aligned} \tag{52}$$

The normalizability of the C-QS wavefunction $\psi_c(r)$ puts restriction on the possible range of values for c_c. In this case, the only possible values of α and c_c^2 which ensure the normalizability of $\psi_c(r)$ is found to be $\alpha > 0$ and $c_c^2 > 0$, implying $c_c > n, n = 1, 2, 3, ...$

The C-QS is now characterized by the following n-independent potential, energy eigenvalues

$$V_c(r) = K_c^2 \left(\left\{ 4c_c^2 - \frac{1}{4} \right\} \tanh^2 K_c r + \frac{3}{4} (\sinh K_c r)^{-2} \right) \tag{53}$$

$$E_n^c = -K_c^2 \left[4(c_c - n)^2 - \left(4c_c^2 - \frac{1}{4} \right) \right] \tag{54}$$

and

$$\Delta E_n^c = 8K_c^2 [c_c - (n + 1/2)].$$

This energy level spacing implies the more stringent inequality $c_c > n + 1/2$, $n = 1, 2, 3, ...$

The corresponding normalized energy eigenfunctions are

$$\begin{aligned} \psi_c(r) = \frac{N_c}{r} & \frac{1}{(\tanh K_c r)^{\frac{1}{2}}} \operatorname{sech}^{2(c_c - n)} K_c r \\ & \times {}_2F_1(-n, 2c_c - n, 2c_c - 2n + 1; \operatorname{sech}^2 K_c r) \end{aligned} \tag{55}$$

where

$$N_c = \frac{2\Gamma(2c_c - n)}{\Gamma(2c_c - 2n + 1)\Gamma(n + 1)} \left(n(c_c - n)(2c_c - n) \right)^{\frac{1}{2}}. \tag{56}$$

This potential of the normal quantum system is similar to the Morse potential [15]. The generated C-QS is similar to the Morse-QS except that C-QS potential has a much steeper repulsive core. They form isospectral QSs modulo a constant energy difference.

6 Discussion

This paper is concerned with the generation of new completely solved quantum bound state systems in the non-relativistic regime, by a transformation method. Here the Hulthen QS (A-system), whose potential is non-powerlaw and whose analytic bound state solutions for s-state only are available, is taken as a typical representative of QSs with non-powerlaw potentials. The first-order application of the transformation method to the Hulthen QS leads to a non-factorizable transformation function $g_B(r)$, as discussed in Section 4. Non-factorizable transformation functions appear whenever the transformation method is applied on a QS with non-powerlaw potential. This furnishes the special type of energy-dependent non-powerlaw potential (Sturmian). These energy-dependent potentials, unlike the usual QSs are always equipped with only a single normalized state, as varying n we do not get excited states, instead we get different QSs. Thus n is no longer a quantum number. It plays the role of a cardinal number parameter to enumerate different QSs. We discuss a method that regroups the set of n-dependent QSs and resurrects n as a quantum number with $(n_{\max} - 1)$ excited states, where $n_{\max}^2 \leq V_0$, thus reproducing a normal QS with n-independent potential. It is evident that the Extended Transformation may be applied any number of times successively to generate new quantum systems. But the important point to be noted is that to generate the C-system starting from the A-system we have to pass through the B-system. Extended Transformations therefore do not form a transformation group.

The first-order application of the transformation method to the Hulthen QS produces a potential similar to the Poschl-Teller potential with finite number of eigenstates. The shape of the potential of this B-QS changes with the value of α, which is symmetrical about $sx = \pi/4$ for $\alpha > 0.25$ and hole becomes oblique with its minimum shifting towards smaller x values as α value increases. For $\alpha = 0.25$ the potential hole becomes symmetrical about $sx = \pi/2$.

The Poschl-Teller potential also behaves in the similar way, *i.e.*, its potential is symmetrical about $sx = \pi/4$ for $\chi = \lambda$. If $\chi > \lambda$, the hole becomes oblique with its minimum shifting to larger x values. But for $1 < \chi < \lambda$, the minimum shifts to the other side as our B-system potential. So applying the transformation method to the Hulthen problem we obtain a potential similar to the Poschl-Teller type with composite parameter α.

The second-order application of the transformation method to the Hulthen system produces C-QS whose potential resembles that of the Morse potential with a steeper potential surface.

From the multiterm potential of the B-system, if we choose $3\eta^2 \csc^2 2\eta g_c(r)$ as the working potential we can generate another new system with normalized energy eigenfunctions and energy eigenvalues. On the other hand if we choose $4\eta^2\alpha^2 \tan^2 \eta g_c(r)$ as the working potential the functional form of $g_c(r)$ is $(1/\eta)\sec^{-1}(\exp(Kr))$, where $K = \sqrt{E_n^c}/2\alpha$. Using this functional form of $g_c(r)$ from equation (58) we can go back to the original Hulthen system by reverting back the n-dependence to α, when β becomes a n-independent constant *i.e.*, undoing the regrouping procedure. In a similar way, taking $K_c^2(4c_c^2 - 1/4)\operatorname{sech}^2 K_c g_D$ as the working potential we find the functional form $g_D(r)$ to be $g_D(r) = (1/K_c)\sinh^{-1}(\tan \rho r)$, where $\rho = \pm\sqrt{E/(4c_c^2 - 1/4)}$ and using this functional form of $g_D(r)$ we can go back to the B-QS by substituting the value of the characteristic constant c_c and keeping α n-independent constant. This shows that although we can generate different QSs by choosing different terms of the multiterm potential as the working potential, out of these, one of the choices will revert it back to the parent QS.

Furthermore we emphasize the fact that the transformation method can be applied, in principle, to generate innumerable exactly solved non-relativistic QSs with non-powerlaw potentials. A subset of the QSs so generated would remain Sturmian QSs, as there is no general procedure that can be applied to render them normal QS. We may mention that the ET method of generation of solvable potential cannot be applied to a QS when the potential is either infinite and/or zero. Such a situation occurs for instance, in case of the quantum mechanical problem of a particle in a box with impenetrable walls. SUSY quantum mechanics can be used in such a situation to produce other isospectral potentials. The methodology involved in ET and SUSY methods are entirely different. From this point of view they may be considered complementary to each other. However, ET can be applied to SUSY quantum mechanical problems also to obtain other solvable SUSY QSs. Work is presently going on in this direction and the results will be published elsewhere.

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